

Parity symmetry and parity breaking in the quantum Rabi model with addition of Ising interaction

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We explore the possibility to generate new parity symmetry in the quantum Rabi model after a bias is introduced. In contrast to a mathematical treatment in a previous publication [J. Phys. A **46**, 265302 (2013)], we consider a physically realistic method by involving an additional spin into the quantum Rabi model to couple with the original spin by an Ising interaction. The rule can be found that the parity symmetry is broken by introducing a bias and then restored by adding new degrees of freedom. Experimental feasibility of realizing the models under discussion is investigated.

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Introduction.— As one of milestones in the history of quantum physics, the well-known quantum Rabi model (QRM) [1] strictly describes the simplest interaction between matter and the quantum light, which has been widely employed to study great variety of physical systems, such as trapped ions [2], cavity and circuit quantum electrodynamics [3–5] as well as photonic systems [6].

Recently, much attention has been paid to the QRM for seeking the closed-form analytical solution and the intrinsic characteristic [7–13]. Although the discrete Z_2 symmetry in the QRM makes the excitation number no longer as a conserved quantity, we are still able to take the QRM as an integrable system by considering the parity conservation [10]. Due to this fact, a parity chain in the QRM has been found in two infinite-dimensional Hilbert invariant subspaces [14], which could be further extended to the N -state case [15].

However, the situation turns to be completely different if a biased field is introduced into the QRM, and we may call it as a biased Rabi model (BRM) with the following form in units of $\hbar = 1$,

$$H_B = -\Delta\sigma^x + \varepsilon\sigma^z + \omega a^\dagger a + \lambda(a^\dagger + a)\sigma^z, \quad (1)$$

where Δ and ε are the tunneling and the local bias field, respectively, ω and a^\dagger (a) are frequency and the creation (annihilation) operator of the single-mode bosonic field, and λ is the Rabi frequency. $\sigma^{z,x}$ are the usual Pauli operators for the spin-1/2 and $\sigma^x = \sigma^+ + \sigma^-$ with $\sigma^\pm = (\sigma^x \pm i\sigma^y)/2$. Please note that the QRM can be described by various Hamiltonians, e.g., unitarily transforming Eq. (1) by $U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$ [10]. But we work throughout this paper by taking the quantization axis defined as in Eq. (1).

Compared to the standard form of the QRM, the additional bias term $\varepsilon\sigma^z$ in Eq.(1) brings in complication but more physics. For example, the parity symmetry in the QRM is broken due to the addition of this bias

[10, 16]. If we define a parity operator $P_1 = \sigma^x \otimes P_0$ with $P_0 = e^{i\pi a^\dagger a}$, we may find $[P_1, H_B] \neq 0$. Although this parity breaking in the BRM can present us some interesting physics, such as observation of unique scaling behavior and further understanding of the rotating-wave approximation [16], it is natural for us to think of the possibility of restoring the broken symmetry. We have noticed a very recent proposal [17] for a new nonlocal symmetry in the BRM, implying a generalized parity. The main idea is the introduction of a transformation P , enabling $\varepsilon \rightarrow -\varepsilon$, $\sigma^z \rightarrow -\sigma^z$ and $a(a^\dagger) \rightarrow -a(-a^\dagger)$. Although it really commutes with H_B , P is not a physically meaningful operator to the BRM as described in [17] because it requires an additional degree of freedom to be involved to carry out $\varepsilon \rightarrow -\varepsilon$ and also lacks concrete models for justification.

In the present work, we focus on a physical consideration of a new symmetry in the BRM. The key idea is the involvement of an additional spin coupled with the original spin by an Ising interaction. This restoration of parity symmetry can be straightforwardly extended to more spins once the new symmetry is also broken by an additional bias. We will discuss the experimental feasibility of demonstrating the new parity symmetry and the parity breaking in the QRM plus Ising model.

New parity symmetry and parity breaking.— By introducing an auxiliary spin into the QRM, we have

$$H_2 = -\Delta \sum_{i=1}^2 \sigma_i^x + \omega a^\dagger a + \lambda(a^\dagger + a)\sigma_1^z + \varepsilon\sigma_1^z\sigma_2^z, \quad (2)$$

where $\sigma_2^{x,z}$ are the Pauli operators for the new spin coupled to the original spin by Ising coupling. In this case, ε is the Ising coupling strength, rather than the bias strength. For convenience of description in the following, we will mention the original spin as the first spin in order to distinguish from the newly joined spins. We may define a new parity operator $P_2 = \prod_{i=1}^2 \sigma_i^x \otimes P_0$,

which fulfills $[H_2, P_2] = 0$. The key point for the physical feasibility of the new symmetry lies in the fact that $P_2^\dagger(\varepsilon\sigma_2^z)P_2 = -\varepsilon\sigma_2^z$, rather than simply making $\varepsilon \rightarrow -\varepsilon$ in [17]. In other words, the new parity for Eq. (1) works only when a new degree of freedom is introduced.

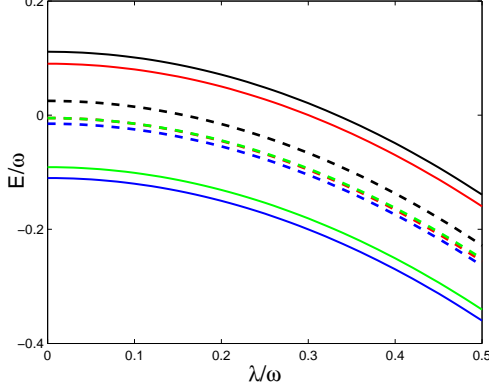


FIG. 1: (Color online) The four lowest eigen-energies of H_2 (dashed, $\eta = 0$) and H_{2B} (solid, $\eta \neq 0$) as functions of the coupling strength λ in the case of $\Delta/\omega = 0.01$, $\varepsilon/\omega = 0.005$ and $\eta/\omega = 0.1$.

The new symmetry will also break down if we introduce a new local bias on the first spin, such as $\eta\sigma_1^z$. In such a case, H_2 turns to be H_{2B} with

$$H_{2B} = -\Delta \sum_{i=1}^2 \sigma_i^x + \omega a^\dagger a + \lambda(a^\dagger + a)\sigma_1^z + \eta\sigma_1^z + \varepsilon\sigma_1^z\sigma_2^z, \quad (3)$$

and it is evident that $[H_{2B}, P_2] \neq 0$. As known in [16], scaling behavior would appear at the critical point of the parity breaking. To see the scaling behavior in the QRM plus Ising model, we may diagonalize Eq. (3) by displaced Fock states $|n\rangle_A = \frac{e^{-q^2/2}}{\sqrt{n!}}(a^\dagger + q)^n e^{-qa^\dagger}|0\rangle$ and $|n\rangle_B = \frac{e^{-q^2/2}}{\sqrt{n!}}(a^\dagger - q)^n e^{qa^\dagger}|0\rangle$ with the displacement variable $q = \lambda/\omega$ [9, 16]. As a result, the eigenfunction of H_{2B} is given by

$$|\Psi\rangle = |\downarrow\downarrow\rangle|\Phi_1\rangle + |\downarrow\uparrow\rangle|\Phi_2\rangle + |\uparrow\downarrow\rangle|\Phi_3\rangle + |\uparrow\uparrow\rangle|\Phi_4\rangle, \quad (4)$$

where $\sigma^z|\uparrow\rangle(|\downarrow\rangle) = |\uparrow\rangle(-|\downarrow\rangle)$, $|\Phi_1\rangle = \sum_n a_n|n\rangle_B$, $|\Phi_2\rangle = \sum_n b_n|n\rangle_B$, $|\Phi_3\rangle = \sum_n c_n|n\rangle_A$ and $|\Phi_4\rangle = \sum_n d_n|n\rangle_A$, with the coefficients a_n , b_n , c_n and d_n to be determined by later calculation. So we have to solve

following Schrödinger equations

$$\begin{aligned} \sum_n (-1)^m D_{mn}(\varepsilon d_n - \Delta c_n) + [\omega(m - q^2) - \eta]a_m \\ - \Delta b_m &= E a_m, \\ \sum_n (-1)^m D_{mn}(\varepsilon c_n - \Delta d_n) + [\omega(m - q^2) - \eta]b_m \\ - \Delta a_m &= E b_m, \\ \sum_n (-1)^n D_{mn}(\varepsilon b_n - \Delta a_n) + [\omega(m - q^2) + \eta]c_m \\ - \Delta d_m &= E c_m, \\ \sum_n (-1)^n D_{mn}(\varepsilon a_n - \Delta b_n) + [\omega(m - q^2) + \eta]d_m \\ - \Delta c_m &= E d_m, \end{aligned}$$

with E the eigenenergy and $D_{m,n}$ defined as [9, 16]

$$D_{m,n} = e^{-2q^2} \sum_{k=0}^{\min[m,n]} (-1)^k \frac{\sqrt{m!n!}(2q)^{m+n-2k}}{(m-k)!(n-k)!k!}.$$

We may obtain analytical solutions of the eigenenergies from above equations under the condition of $\Delta/\omega \ll 1$ [18], for which the diagonal terms of $D_{m,n}$ play dominant roles with respect to the off-diagonal terms. In such a case, the eigenenergies are given by $E_{m1}^\pm = -\Delta \pm \sqrt{D_{mm}^2(\varepsilon - \Delta)^2 + \eta^2 + \omega(m - q^2)}$, and $E_{m2}^\pm = \Delta \pm \sqrt{D_{mm}^2(\varepsilon + \Delta)^2 + \eta^2 + \omega(m - q^2)}$, with $m = 0, 1, \dots$, where E_{01}^- is the ground-state eigenenergy. Fig. 1 plots the lowest four eigenenergies, from which we know that the introduction of the bias into H_2 , breaking down the the parity of the original system, shifts the eigenenergies in a symmetric way. i.e., half of the eigenenergies being lower and half being higher. As a result, the ground-state eigenenergy is lower after the bias is introduced.

Scaling behavior.— Using the ground-state eigenfunction, we obtain,

$$\langle\sigma_1^z\rangle = \frac{-\kappa}{\sqrt{\kappa^2 + e^{-4\beta}}}, \quad (5)$$

with $\beta = q^2$ and $\kappa = \eta/(\Delta - \varepsilon)$. Compared with the relevant results in [16], the Ising coupling strength ε is involved in κ , which would definitely modify the scaling behavior. Following the steps in [16], we define a scale $\beta_c = -\ln(2\kappa^2)/4$ and a displaced scale $\alpha = (\beta - \beta_c)/\sqrt{27}$, and then we have

$$\langle\sigma_1^z\rangle = \frac{-\kappa}{\sqrt{\kappa^2 + (2\kappa^2)^{\beta/\beta_c}}}, \quad (6)$$

and

$$\langle\sigma_1^z\rangle = -1/\sqrt{1 + 2e^{-12\sqrt{3}\alpha}}, \quad (7)$$

the latter of which is independent of κ and shows scaling invariance.

For a fixed value of κ , $\langle \sigma_1^z \rangle$ in Eq. (6) is only relevant to the variable β , rather than to other characteristic parameters. So β_c can be regarded as a scale of the QRM. Different from in [16], however, the added Ising coupling leads to a bifurcation in the scaling behavior, as shown in Fig. 2(a) where the lower (upper) branch corresponds to $\Delta > \varepsilon$ ($\Delta < \varepsilon$). In addition, $\beta = \beta_c$ corresponds to fixed crossing points with the variation of β , in which $\langle \sigma_1^z \rangle$ turns out to be constants $\pm 1/\sqrt{3}$, i.e., the fixed crossing points existing in the two branches. After a scaling displacement, Eq. (7) is of the same form as in [16] and the variation with α is formally independent of κ . As shown in Fig. 2(b), the effect of the Ising coupling is reflected in different values of α and $\langle \sigma_1^z \rangle$ in the curve.

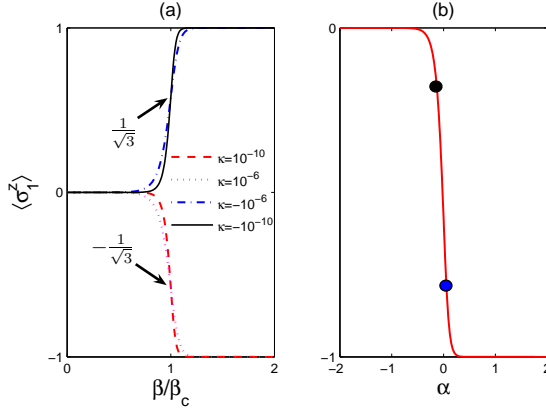


FIG. 2: (Color online) Scaling behavior of the ground-state $\langle \sigma_1^z \rangle$. (a) As a function of β/β_c , which shows a bifurcation depending on the difference between Δ and ε ; (b) As a function of α , which remains unchanged with respect to different values of κ . The black (upper) dot means $\eta/\Delta = 10^{-6}$ and $\varepsilon = 0$, and the blue (lower) dot represents $\eta/\Delta = 10^{-6}$ and $\varepsilon/\Delta = 0.5$ in the case of $q = 0.2$.

Discussion.- Our treatment above can be generalized to the N -spin case with one spin under QRM and coupled to other $N - 1$ spins by Ising interactions in a star configuration, which is given by

$$H_N = -\Delta \sum_{i=1}^N \sigma_i^x + \omega a^\dagger a + \lambda(a^\dagger + a)\sigma_1^z + \sigma_1^z \otimes \sum_{k=2}^N \varepsilon_k \sigma_k^z, \quad (8)$$

where ε_k is the Ising coupling strength of the first spin with the k th one. H_N possesses a parity symmetry with the corresponding parity operator $P_N = \prod_{i=1}^N \sigma_i^x \otimes P_0$, due to $[H_N, P_N] = 0$. As described above, this symmetry will be broken by an additional local bias on the first spin, such as $\eta\sigma_1^z$, and the parity breaking leads to scaling behavior of the ground state similar to Eq. (5), but with $\kappa = \eta/(\Delta - \sum_{k=2}^N \varepsilon_k)$ in the present case. Evidently, a new parity symmetry will appear once a new spin moves

in and turns the bias to be an Ising coupling to the first spin. Simply speaking, it is a rule that the parity symmetry and breaking appear alternately in the QRM by introducing a term of Ising interaction and a term of bias.

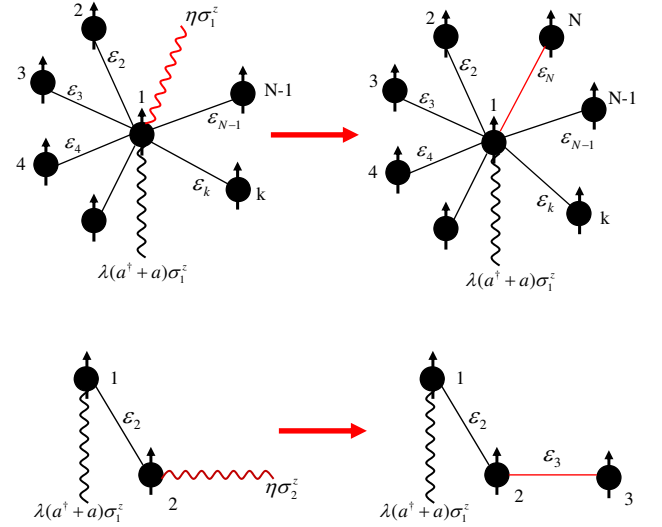


FIG. 3: (Color online) Upper panel: Sketch for star configuration, where the bias is applied on the first spin to break down the parity symmetry and a new spin is introduced to generate a new parity symmetry by turning the bias to be an Ising interaction with the first spin; Lower panel: Sketch for linear configuration, where the bias is applied on the newly joined spin (e.g., the second spin) to break down the parity symmetry and an additional spin is introduced to generate a new parity symmetry by turning the bias to be an Ising interaction with the second spin.

However, what we described above is for a multi-spin Ising coupling in a star configuration (See Fig. 3(a)), where the newly introduced spins only couple to the first spin, and no coupling between any two of the newly joined spins is assumed. What about other configurations, such as the linear structure in Fig. 3(b) with the bias applied on the newly joined spin? We find by straightforward deduction that the rule above still works in this case, and the scaling behavior relevant to the parity breaking is also observable if we measure $\langle \sigma_1^z \rangle$. The key point for observing the scaling behavior is that our measurement should be made on the spin coupling directly to the quantized field under the QRM. For other spins without direct couplings to the QRM field, no scaling behavior can be observed on them in the parity breaking.

Since both QRM and Ising model are usually employed interactions in different fields of physics, we may achieve the models described above and observe the predicted behavior with current laboratory technique. Taking the ion-trap system as an example, we first consider a single ultracold ion confined in the pseudo-potential of a Paul trap under laser irradiation in a Raman Λ -type configu-

ration, whose hamiltonian in a frame rotating with the laser frequency is given by [7],

$$H_{ion} = \frac{\tilde{\Delta}}{2}\sigma^z + \tilde{\nu}a^\dagger a + \frac{\tilde{\Omega}}{2}[\sigma^+ e^{i\tilde{\eta}(a^\dagger + a)} + \sigma^- e^{-i\tilde{\eta}(a^\dagger + a)}], \quad (9)$$

where $\tilde{\Delta}$ is the detuning of the laser to the two levels of the spin, $\tilde{\nu}$ is the trap frequency with $a^\dagger(a)$ the creation (annihilation) operator of the vibrational mode and $\sigma^{z,\pm}$ are the usual Pauli operators for the spin. $\tilde{\Omega}$ is the Rabi frequency and $\tilde{\eta}$ is the Lamb-Dicke parameter. As shown in [7, 16], H_{ion} can turn into a similar form to Eq. (1) after some unitary transformation, where the bias is relevant to the detuning $\tilde{\Delta}$. To achieve the Ising model, we introduce another ion coupling to the first ion as,

$$H_{cc} = \omega_s S^z + \tilde{\epsilon} \sigma^z S^x, \quad (10)$$

where $S^{z,x}$ are the usual Pauli operators for the new spin, ω_s is the splitting frequency of the new spin with coupling strength $\tilde{\epsilon}$ to the first one. The coupling $\sigma^z S^x$ and similar forms of Ising coupling have been achieved experimentally by off-resonant lasers and resonant Raman beams in trapped-ion systems [19–22]. Such couplings can also be generated by a magnetic field gradient [23] or by a non-uniform laser field [24] on the trapped ions. Following the unitary transformations in [7] and meanwhile performing a Hadamard gate on the new spin for $S^z \Leftrightarrow S^x$, we may reach

$$H'_{ion} = -\frac{\tilde{\Omega}}{2}\sigma^x + \omega_s S^x + \tilde{\nu}a^\dagger a + \frac{\tilde{\nu}\tilde{\eta}}{2}(a^\dagger + a)\sigma^z - \frac{\tilde{\Delta}}{2}\sigma^z + \tilde{\epsilon}\sigma^z S^z,$$

which is of the same form as in Eq. (3). So the parity symmetry and the parity breaking can be achieved experimentally by tuning $\tilde{\Delta} = 0$ and $\tilde{\Delta} \neq 0$, respectively.

The models under consideration are also feasible in circuit QED systems [25–27] and optomechanical system [28], as exemplified in [16], by introducing an auxiliary spin coupling to the first spin by Ising interaction. A previous publication has shown a quantum nondemolition detection by an auxiliary spin through such an Ising coupling for light-matter interaction in a superconducting system [29].

Conclusion.- We have investigated the parity symmetry and parity breaking relevant to the QRM by considering involvement of new spins. We obtained a general rule and also explored the scaling behavior occurring in the case of the parity breaking, which strongly depends on the Ising interaction. The experimental feasibility to demonstrate the models and the unique behavior is discussed. We believe that our results would be helpful for further understanding light-matter interaction.

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